Research project

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I am interested by rigidity phenomena in locally conformally symplectic (LCS) geometry, which is a generalization of symplectic geometry in which transition maps may only preserve the canonical symplectic form on $\mathbb{R}^{2n}$ up to some positive constant factor. This generalization shares the local properties of symplectic geometry (allowing, for example, for the study of Hamiltonian dynamics). This allows us to define (locally confromally) exact symplectic structures as well as (locally conformally) exact Lagrangians, which are the objects I have studied during my thesis.

Broadly, I intend to continue the overarching goal of my thesis of trying to adapt results from symplectic geometry to LCS geometry. Therefore in the short term, I intend to pursue three venues of research: adapting (derived) sheaf theory, Hamiltonian-Floer homology, and Lagrangian-Floer homology. I would also like to showcase the usefulness of those adaptations by proving an LCS version of the Abouzaid-Kragh theorem. I am also interested in the interactions between LCS and contact geometry, especially I would like to further study the similarities between Reeb chords and essential Liouville chords.

1. Past research

More ample details about my past research are freely available on hal.science in my thesis' manuscript.

To sum up my past research, I've been interested in the creation of tools to study rigidity phenomena in LCS geometry. Broadly, there are two main topics I've researched : the parallels between contact and LCS geometry (especially, the relationship between Reeb chords and the so-called essential Liouville chords) and the study of generating function (especially, proof strategies that could be adapted to Floer homologies and derived sheaf theory). If I had to highlight three main results, it would be the following.

Proposition 1. *For any n, there is a closed manifold M and a closed exact Lagrangian (of LCS type) $L \subset T^*M$ such that the canonical projection of L on M is of degree n. Moreover, there are closed manifolds M and closed exact Lagrangians (of LCS type) $L \subset T^*M$ such that the Euler characteristic of L and M differ. However, for any closed manifold M and any closed exact Lagrangian (of LCS type) $L \subset T^*M$, if $rank(H^1(M)) \geq 1$, then $rank(H^1(L)) \geq 1$.*

Contrasting the proposition with the Abouzaid-Kragh theorem, this illustrates that the topology of exact Lagrangian (of LCS type) is vastly more complicated than that of exact Lagrangians (of symplectic type) but is not completely random.

Theorem 1. *Given a closed manifold M and a closed exact Lagrangian (of LCS type) $L \subset T^*M$, under some generic conditions on L and some technical condition to allow us to define essential Liouville chords, if L has no essential Liouville chord, then the canonical projection of L on m is a simple homotopy equivalence*

Swapping out the LCS language for the "contact geometry" language (substitute exact Lagrangian with Legendrian, cotangent bundle with jet bundle and exact Liouville chord with Reeb chord) illustrates the parallels between Reeb chords and Liouville chords. Moreover, given a Legendrian in J^1M, it can be lifted to a closed exact Lagrangian (of LCS type) in $T^*(M \times \mathbb{S}^1)$ and the Reeb chords will lift (in some fashion) to essential Liouville chords. While those parallels run deeper than just this theorem, this should be enough to illustrate the similarities between contact and LCS geometry.

Finally, the last result I want to share is a refinement of a result by Chantraine and Murphy found in [CM16]. However, my strategy of proof for this result is completely different and this showcases the usefulness of this new strategy.

Theorem 2. *Let M be a closed manifold, $\beta \in \Omega^1(M)$ be closed (we will also call β the various pullbacks of β) and $F : M \times \mathbb{R}^k \to \mathbb{R}$ be a smooth function that is a quadratic form Q on R^k outside of a compact set, and such that $dF - F\beta$ intersects the 0-section transversely. We will write $Crit^\beta(F)$ the set of those intersection points. Since $dF - F\beta$ is locally the differential of a Morse function (up to a conformal factor), we can canonically define the index of the intersection points. We will write $Crit_i^\beta(F)$ the set intersection points of index i. Let p the dimension of the subvectorspace on which Q is negative definite. Then*

$$\#Crit_i^\beta(F) \geq rank(HN_{i-p}(M, \beta))$$

were $HN_*(M, \beta)$ denotes the Morse-Novikov homology of M with respect to β.

2. Derived sheaves

This is a very near future project as most of the work is done, and as such may well be completed before the start of this post-doc.

Derived sheaf theory has been successfully used by Guillermou, Kashiwara and Shapira in [GKS12] to show Laudenbach-Sikorav's theorem (a symplectic version of Chantraine-Murphy's theorem). Here, I use the insights derived from my research on Morse-Novikov homology to find a category of sheaves useful for the study of exact Lagrangians. Taking inspiration in [GKS12], I will then use my new definitions to give a sheaf-theoretical proof of the Chantraine-Murphy theorem, thus showcasing the usefulness of this category of sheaves.

The main quantity that is studied to establish this result is what I call the asymptotic Betti number. It can be defined as follow. Given M a closed connected manifold, $\beta \in \Omega^1(M)$ closed and λ the canonical Liouville form on T^*M, call M_β the integral cover of β and $\pi : M_\beta \to M$ the canonical projection. Then the sheaves of interest are sheaves of the form $G = \pi^{-1}F$ for $F \in D^b(\mathbb{R}_M)$ an $\mathbb{R}$-constructible sheaf. Given W_k one of the "good" compact exhaustions mentionned in my thesis, we can define the asymptotic Betti number of the sheaf

$$c_i(F) = \frac{b_i(G_{W_k - \partial_- W_k})}{(2k+1)^r},$$

where r is the minimal number of element necessary to generate the group of deck transformations. This number does not depend on the good compact exhaustion chosen.

3. Pseudo-holomorphic curves and Floer homologies

Here again, this part of my research project relies on this good compact exhaustion and more generally on the ideas I have developed during my thesis. First, I am interested in creating an Hamiltonian-Floer homology for LCS geometry. This part will rely heavily on Groman's work on Floer theory for open manifolds (see [Gro15]). Given an Hamiltonian, it should a priori be possible to combine his work and my thesis to find a good sequence of lower semi-continuous functions such that the limit of the Hamiltonian-Floer homology groups yields a group containing some relevant data. For now, this project is mainly concerned by LCS structures defined on cotangent bundles.

Moreover, I am interested in adapting Lagrangian-Floer homology to LCS geometry. The exact course for this is as of now a bit obscure as adaptations tend to run afoul of Gromov's compactness (see [CM16] for a short discussion on the subject), but Fabio Gironella, Baptiste Chantraine and I are currently trying to figure out how insights gained from my thesis could be applied to this problem.

Finally, I plan on applying the insights from adapting the Floer homologies to try and adapt Gromov-Witten theory to this new setting. Once done, I would be interested in exploring the ramifications of this adaptation to low-dimensional topology.

4. An adaptation of the Abouzaid-Kragh theorem

The Abouzaid-Kragh theorem was re-proven (mostly) by S. Guillermou in [Gui19] using derived sheaves. As such, and since a Lagrangian-Floer homology for LCS geometry is a bit nebulous as of yet, this part of the research project relies heavily on his work for now.

4.1. On the projection of exact quantifiable Lagrangians

As discussed in the "Past research", there are some exact Lagrangians whose Morse-Novikov homology is not that of the base manifold (never even mind being isomorphic through the projection). Therefore, any LCS adaptation of the theorem should take into account those exceptions.

Definition 1. Let M be a closed manifold and take some closed $\beta \in \Omega^1(M)$. We will β the various pullbacks of β. Let $i : L \to T^*M$ be a β-exact Lagrangian embedding in (T^*M, λ, β) such that $i^*\lambda = d_\beta f$.Then the pre-conification of L is given by

$$\begin{aligned} j : L \times \mathbb{R}_+^* &\to T^*(M \times \mathbb{R}) \\ (l, t) &\mapsto (i_1(l), -ti_2(l) - f\beta, f(l), t) \end{aligned}$$

We will note $\tilde{C}_\beta(L) := j(L \times \mathbb{R}_+^*)$, and $C_\beta(L)$ will be the preimage of $\tilde{C}_\beta(L)$ in the integral cover of β. This last submanifold will be called the conification of L

We will day that L is quantifiable if and only if there is some simple β-sheaf F such that $SS(F) \cap T^*M - M = C_\beta(L)$.

Now, it is not quite certain that this is the good the definition of quantifiable Lagrangian, although the true definition if there is one should not be too far from this. However, we wish to prove the following conjecture:

Conjecture 1. *(Project) Let L be a β-exact quantifiable Lagrangian of (T^*M, λ, β) and π : $T^*M \to M$ the projection, then :*

$$\pi_* : HN_*(L, i^*\beta) \xrightarrow{\sim} HN_*(M, \beta)$$

Do note that it is not quite clear at this point how we could get back the torsion subgroups of Morse-Novikov homology from β-sheaves, if it can be done at all. However, any good notion of quantifiable should at the very least satisfy this following conjecture:

Conjecture 2. *(Project, weak version) Let L be a β-exact quantifiable Lagrangian of (T^*M, λ, β) and $\pi : T^*M \to M$ the projection, then :*

$$\pi_* : HN_*(L, i^*\beta)/Tors \xrightarrow{\sim} HN_*(M, \beta)/Tors$$

Do note that if any of the conjectures is true, then some exact Lagrangians will not be quantifiable.

Now, there will be quite a couple of differences with Guillermou's approach to the proof of this theorem. Not only are the manifolds non-compact, but we wish for more than just the cohomology of some skyscraper sheaves on the total manifold. Moreover, while a good proof might establish some isomorphisms between β-sheaves, it might not necessarily be quite clear how to get an isomorphsim on the Morse-Novikov homologies.

4.2. Quantification of exact Lagrangians

Now, whichever definition of quantifiable makes the above work, we will want to figure out the conditions that we have to put on an exact Lagrangian for it to be quantifiable. Classical proof consists on taking a double of a 0-exact Lagrangian, and then building a sheaf that quantifies that. One of the doubles is then pushed to infinity. This would indeed provide some obstacle to the quantifiability of exact Lagrangians, as the double can not always be pushed away.

As such, it is the author's hope to find the following result:

Conjecture 3. *(Project) Let L be a β-exact quantifiable Lagrangian of (T^*M, λ, β). Then the double of L is quantifiable and, if one of the copies can be pushed to infinity through an Hamiltonian isotopy ϕ_t of $C_\beta(L)$ such that for every $t > 0$, $\phi_t(C_\beta(L)) \cap C_\beta(L) = \varnothing$, we have that L is quantifiable*

It might be however necessary to strengthen the conditions on L. An example of that would be:

Conjecture 4. *(Project, weak version) Let L be a β-exact quantifiable Lagrangian of (T^*M, λ, β) such that $C_\beta(L)$ is connected. Then the double of L is quantifiable and, if one of the copies can be pushed to infinity through an Hamiltonian isotopy ϕ_t of $C_\beta(L)$ such that for every $t > 0$, $\phi_t(C_\beta(L)) \cap C_\beta(L) = \varnothing$, we have that L is quantifiable*

Here, problems are mostly linked to compacity. However, do note that every object here has some kind of invariance with respect to deck transformations.

5. References

[CM16] Baptiste Chantraine and Emmy Murphy. "Conformal symplectic geometry of cotangent bundles". In: *Journal of Symplectic Geometry* (2016).

[GKS12] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. "Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems". In: *Duke Mathematical Journal* 161.2 (2012), pp. 201 –245. DOI: 10.1215/00127094-1507367.

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